Implicit Differentiation 05/12/2023

Themes: What is implicit differentiation?
Application: Related rates.

[1] For the curve defined by $x^4 + y^3 + axy + b = 0$, the derivative $\dfrac{dy}{dx}$ at the point $(0, 1)$ is -2. Determine the product ab.

[2] Consider the curve defined by the equation $\sqrt[3]{x} + a\sqrt[3]{y} = 29$. At the point $(8, b)$ on this curve, the slope of the tangent line is $\dfrac{27}{4}$. Determine the values of the constants a and b.

[3] Consider the parametric equations $x = \dfrac{1-t^2}{1+t^2}$ and $y = \dfrac{2t}{1+t^2}$. If $t = 2$, what is the value of the derivative $\dfrac{dy}{dx}$?

[4] A 5 m ladder is leaning against a vertical wall. Suppose that the bottom of the ladder is being pulled away from the wall at a rate of 1 m/s. How fast is the area of the triangle underneath the ladder changing at the instant that the top of the ladder is 4 m from the floor?

[5] *Extension question*: A lighthouse containing a revolving beacon is located 3 km from P, the nearest point on a straight shoreline. The beacon revolves at a constant rate of 4 revolutions per minute and throws a spot of light onto the shoreline. How fast is the spot of light moving when it is

(a) at P?

(b) at a point on the shoreline 2 km from P?

Note: $\dfrac{d}{d\theta} \tan\theta = \sec^2\theta$

Suggested solutions: Implicit Differentiation 05/12/2023

[1] The given curve passes through the point $(0, 1)$. From the equation $1 + b = 0$, it follows that $b = -1$.

Next, we implicitly differentiate the given curve's equation:

$$4x^3 + 3y^2\frac{dy}{dx} + ay + ax\frac{dy}{dx} = 0.$$

Rearranging terms, we get:

$$\frac{dy}{dx}(3y^2 + ax) = -4x^3 - ay.$$

This simplifies to:

$$\frac{dy}{dx} = -\frac{4x^3 + ay}{3y^2 + ax}.$$

We are given that the slope $\frac{dy}{dx}$ is -2 at the point $(1, 0)$. Substituting these values into the equation, we get:

$$-2 = -\frac{4(1)^3 + a(0)}{3(0)^2 + a(1)}.$$

Simplifying, this leads to $-2 = -\frac{4}{a}$, and hence $a = 2$.

Finally, since we have $a = 2$ and $b = -1$, the product ab is $(2)(-1) = -2$.

[2] The curve intersects the point $(8, b)$, which implies that $a\sqrt[3]{b} = 27$. Next, we implicitly differentiate with respect to x, yielding

$$\frac{1}{3\sqrt[3]{x^2}} + \frac{a}{3\sqrt[3]{y^2}}\frac{dy}{dx} = 0.$$

This leads to the derivative

$$\frac{dy}{dx} = -\frac{\sqrt[3]{y^2}}{\sqrt[3]{x^2}}.$$

Furthermore, at the point $(8, b)$, the slope is $\frac{27}{4}$. Substituting these values into our derivative equation, we obtain

$$\sqrt[3]{b^2} = -27a.$$

Solving these equations simultaneously gives us the values $a = -3$ and $b = -729$.

[3] By performing parametric differentiation, we obtain the derivatives of x and y with respect to t as follows:

$$\frac{dx}{dt} = \frac{-4t}{(1+t^2)^2},$$
$$\frac{dy}{dt} = \frac{-2t^2+2}{(1+t^2)^2}.$$

To find the derivative $\frac{dy}{dx}$, we use the chain rule, which gives us:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2t^2+2}{-4t} = \frac{t^2-1}{2t}.$$

Therefore, at $t = 2$, the derivative $\frac{dy}{dx}$ is calculated as:

$$\frac{dy}{dx} = \frac{2^2-1}{2 \cdot 2} = \frac{3}{4}.$$

[4] Note that the relation is $x^2 + y^2 = 5^2$
Differentiating both sides with respect to t.

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$
$$2x\frac{dx}{dt} = -2y\frac{dy}{dt}$$

Since $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = -\frac{x}{y}$
At the point where $y = 4, x = \sqrt{25 - 4^2} = 3$
Hence, $\frac{dy}{dt} = -\frac{3}{4}$.
To find the rate of change of the area, consider $A = \frac{1}{2}xy$

Differentiating both sides with respect to t,

$$\begin{aligned}\frac{dA}{dt} &= \frac{1}{2}\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right),\\ &= \frac{1}{2}(3 \times \frac{-3}{4} + 4(1)),\\ &= \frac{-1}{8}.\end{aligned}$$

The area is being reduced by $\frac{1}{8}$ m^2/s.

[5] (a) Construct a triangle with a line with point P and a point off the line L and some other point along the line, and derive $\tan\theta = \frac{x}{3}$. We want to calculate $\frac{dx}{dt}$ when $x = 0$ and $\theta = 0$.
The beacon revolves at 4 rev/min.
So $\frac{d\theta}{dt} = 4\ rev/min = 8\pi radians/min$.
Implicitly differentiating $x = 3\tan\theta$ produces

$$\begin{aligned}\frac{dx}{dt} &= 3\sec^2\theta\frac{d\theta}{dt},\\ \frac{dx}{dt} &= 3\sec^2\theta \times 8\pi,\\ &= 24\pi\ km/min.\end{aligned}$$

(b) When $x = 2, \tan\theta = \frac{2}{3}$.
Using the equation from part (a).

$$\begin{aligned}\frac{dx}{dt} &= 3\sec^2\theta\frac{d\theta}{dt},\\ \frac{dx}{dt} &= 3\sec^2 tan^{-1}(\frac{2}{3}) \times 8\pi,\\ &= 3 \times \frac{13}{9} \times 8\pi,\\ &= \frac{104\pi}{3}\ km/min.\end{aligned}$$